A Linear Consensus Approach to Quality-Fair Video Delivery

Laura Dal Col, Sophie Tarbouriech, Luca Zaccarian, Michel Kieffer

▶ To cite this version:

Laura Dal Col, Sophie Tarbouriech, Luca Zaccarian, Michel Kieffer. A Linear Consensus Approach to Quality-Fair Video Delivery. 53rd IEEE Annual Conference on Decision and Control (CDC 2014), Dec 2014, Los Angeles, CA, USA, United States. pp. 5296-5301, 10.1109/cdc.2014.7040217 . hal-01068057

HAL Id: hal-01068057
https://hal.science/hal-01068057

Submitted on 26 Sep 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A Linear Consensus Approach to Quality-Fair Video Delivery

L. Dal Col, S. Tarbouriech, L. Zaccarian, M. Kieffer

Abstract— **We consider the problem of delivering encoded video to several mobile users sharing a limited wireless resource. The aim is to provide some fairness among the terminals in terms of utility, which is cast in the framework of discrete-time linear distributed consensus. To this end, the rate-utility characteristics of each stream is linearized, which allows to get necessary and sufficient conditions on the controller parameters to asymptotically reach the consensus. To prove our statement we also provide a general result on consensus of identical continuous- or discrete-time linear systems. Simulation results illustrate the effectiveness of the proposed approach.**

I. INTRODUCTION

Multimedia contents delivered to mobile clients represent a growing part of the internet traffic [1]. Even if the development of 4G networks increases the available wireless resources, high-quality video contents are increasingly demanding in terms of transmission rates. When several users share some communication link to get streamed video contents, simple bit-rate or bandwidth fair allocation strategies are usually inappropriate. Such strategies are agnostic of the rate-quality characteristics of the delivered contents. Rather static video contents such as news may be efficiently delivered with a moderate bit rate, that would be insufficient to enjoy an action motion picture of decent quality. This has motivated the recent development of quality-fair video delivery techniques, such as [2], [6], [7], [12].

For example, [6] considers an utility max-min fair resource allocation, which tries to maximize the worst utility. Nevertheless, it does not consider the temporal variability of the rate-utility characteristics of the contents, or the delays introduced by the network and the buffers of the delivery system. In [12], a content-aware distortion-fair video delivery scheme is proposed assuming that the characteristics of video frames are known in advance, which restricts its usage to the streaming of stored videos. In [7], a Lagrangian optimization framework is considered to maximize the sum of the achievable rates while minimizing the distortion difference among streams. This requires to gather all rate-utility characteristics of the streams at the control unit. The user experience is accurately modeled in [5] using the empirical cumulative distribution function of the predicted video quality. This paper also considers admission control and uses constrained optimization techniques, but again, rate-utility characteristics of the videos are required.

Feedback control techniques have been considered in [3], [4] to reach a quality fairness among users, while controlling the level of the buffers in the network or the buffering delay. However, the tuning of the parameters is nontrivial and has been performed heuristically in these works. The main goal of this paper is to systematically select those gains based on a linear consensus approach. Consensus and synchronization problems are extensively studied in the literature for identical multi-agent systems, see, *e.g.*, [17] [8], [9], [26]. In *consensus* problems, the emphasis is on communication constraints, where the individual systems are modeled as simple integrators, and the collective evolution is determined by the exchange of information modeled by some communication graph. It has been shown in [13], [14] that mild assumptions on graph connectivity ensure to uniformly exponentially reach consensus, see also [11] and [16]. Compared to consensus problems, in the *synchronization* literature the focus is primarily in the individual dynamics. As in the consensus problem the objective is to synchronize the system to a common trajectory by exchanging relative information [10], [22], [23]. Recently, some efforts have been made to understand consensus and synchronization problems in which both the individual dynamics as well as communication constraints play an important role [27], [15], [21], [29].

The aim of this paper is to formulate the quality-fair video delivery problem as a distributed consensus problem. Using linearized rate-utility characteristics of the video contents, it is possible to derive necessary and sufficient conditions for the coupled system to converge to a constant consensus state. Optimal control gains may then be chosen to maximize, *e.g.*, the convergence rate. Simulation results show that the control gains are appropriately designed, providing better performance in terms of quality fairness than the reference scheme [3], [4], where a heuristic technique is used for the tuning. Moreover, the dependency in the characteristics of the video contents and in the number of users in the expression of the optimal gains is evidenced.

To prove our results, we derive a fairly general result on continuous- and discrete-time consensus seeking for identical linear systems (given in Theorem 2). As the main contribution in the present work, Theorem 1 gives necessary and sufficient conditions for the uniform global exponential synchronization between the agents. Section II describes the system under consideration. Section III casts the problem as a linear consensus problem and gives the main result. In

L. Zaccarian, S. Tarbouriech and L. Dal Col are with CNRS, LAAS, 7 avenue du colonel Roche, F-31400 Toulouse, France and Univ de Toulouse, LAAS, F-31400 Toulouse, France. L. Zaccarian is also with the Dipartimento di Ing. Industriale, University of Trento, Italy. zaccarian@laas.fr, tarbour@laas.fr, ldalcol@laas.fr

M. Kieffer is with L2S - CNRS - Univ Paris-Sud, 3 rue Joliot-Curie, F-91192 Gif-sur-Yvette, kieffer@lss.supelec.fr. M. Kieffer is partly supported by the Institut Universitaire de France.

Work supported by the ANR project LimICoS contract number 12 BS03 005 01.

Section IV, a way to select suboptimal controller gains is proposed and the effectiveness of this selection is illustrated on experimental tests in Section V. Detailed proofs of these results are posponed in Section VI, whereas Section VII concludes the paper. Appendix I contains a detailed derivation of the results in Lemma 1, while Appendix II provides the proof of Lemma 2.

Notation. We use $x^+ = x^+(j) = x(j+1)$ to denote the push-forward operator, $\forall j \in \mathbb{Z}_+$, $x^d = x^d(j) = x(j-1)$ to denote the one step delay operator, and $x^{dd} = x^{dd}(j) = x(j-2)$ to denote the two steps delay operator.

II. Problem Formulation

In this paper we analyze the model considered in [4] where quality fairness is conjugated as ensuring the same rate-utility value for all video streams. The dynamics of the *i*-th video stream, $i = 1, \ldots, N$, is described by the following set of equations (conveniently reported from [4, equation (22)]):

$$a_i(j)^+ = a_i(j) + \delta a_i(j) \tag{1a}$$
$$a_i^d(j)^+ = a_i(j) \tag{1b}$$
$$\Phi_i(j)^+ = \Phi_i(j) + \Delta U_i^{dd}(j) - U_i^{dd}(j) \tag{1c}$$
$$\Pi_i^b(j)^+ = \Pi_i^b(j) + (B_i(j) - B_0) \tag{1d}$$
$$R_i^{ed}(j)^+ = R_0 - \tfrac{K_P^{eb}+K_I^{eb}}{T}(B_i(j) - B_0) - \tfrac{K_I^{eb}}{T}\Pi_i^b(j) \tag{1e}$$
$$R_i^{edd}(j)^+ = R_i^{ed}(j) \tag{1f}$$
$$U_i^{dd}(j)^+ = f(a_i^d(j), R_i^{ed}(j)) \tag{1g}$$
$$\begin{aligned} B_i^+(j) = B_i(j) + [R_i^{edd}(j) - R_0 + \\ + (K_P^t + K_I^t)\Delta U_i^{dd}(j) - K_I^t \Phi_i(j)]T \end{aligned} \tag{1h}$$
$$\bar{U}^{dd}(j) = \tfrac{1}{N}\sum_{k=1}^{N} U_k^{dd}(j) \tag{1i}$$

The discrete-time nonlinear state-space representation in (1) considers N mobile users, indexed by the subscript i, connected to the same base station (BS) and sharing wireless resources provided by the BS to get streamed videos delivered by N remote servers. Time is assumed to be slotted with a period T. Each video delivery chain is assumed to be controlled in a synchronous way, with video streams consisting of group of pictures (GoP) of the same duration T. Control is performed in a media-aware network element (MANE). The rate-utility function of the j-th GoP of the i-th stream is modeled by a nonlinear function $U_i(j) = f(a_i(j), R)$ parametrized by the vector $a_i(j)$ of the video characteristics and depending on the video encoding rate R. The evolution of $a_i(j)$ is described by (1a), with $\delta a_i(j)$ representing some uncontrolled perturbation modeling the variations with time of the rate-utility characteristics. A total transmission rate R_c is assumed to be shared by the users. The encoding rate target is evaluated within the MANE using an internal PI controller (controller K_{int}) aiming at regulating the buffer level B_i of the i-th stream around some reference buffer level B_0, see (1d) and (1e). K_P^{eb} and K_I^{eb} are the proportional and integral control parameters for the encoding rates. $R_0 = R_c/N$ is the average rate, which would be allocated in a rate-fair scenario. The draining rate of the i-th buffer within the MANE is controlled so as to minimize the discrepancy $\Delta U_i(j)$ of the utility $U_i(j)$ of the i-th program with respect to the average utility given by (1i). For that purpose, an external PI controller (controller K_{ext}) with parameters K_P^t and K_I^t is involved: programs with a utility less than average are drained faster, leading to an increase of the encoding rate. A one-period forward and backward delay between the MANE and the server is considered to account for moderate queuing delays in the network. Provided that T is of the order of the second, this is a realistic upper bound. The delay operators account for these delays in (1).

The problem addressed in this work concerns the selection of the four PI controller gains to ensure the asymptotic convergence of the utilities $U_i(j)$ in (1g) to a common value $\bar{U}$, namely:

$$\lim_{j\to+\infty} U_i(j) = \bar{U}, \quad \forall i = 1, \ldots, N. \tag{2}$$

In Theorem 1, condition (2) is shown to hold if and only if the spectral properties of suitably defined matrices are ensured. This will allow for the optimal gains selection proposed in Section I.

III. Consensus analysis for the linearized dynamics

A. Two PI control loops

System (1) can be rearranged in order to highlight the different contributions of two PI controllers. The first one essentially rejecting the constant bias B_0, and the second one rejecting the constant bias R_0 and inducing consensus of the utilities of the video streams. The first PI controller (denoted by K_{int} in Figure 1) corresponds to an internal loop and is characterized by (1d) and (1e), rewritten as:

$$\Pi_i^{b+} = \Pi_i^b + \Delta B_i \tag{3a}$$
$$\kappa_1 = \tfrac{k_I^{int}}{T}\Pi_i^b + \tfrac{k_P^{int}}{T}\Delta B_i, \tag{3b}$$

where Π_i^b is the controller state, $\Delta B_i = B_i - B_0$ is the controller input and $\kappa_1 = -\Delta R_i^e = -(R_i^e - R_0)$ is the controller output. The integral and proportional gains k_I^{int} and k_P^{int} are defined as:

$$k_I^{int} = K_I^{eb}, \quad k_P^{int} = K_P^{eb} + K_I^{eb}. \tag{4}$$

The second PI controller (denoted by K_{ext} in Figure 1) is characterized by (1c) and (1h), rewritten as:

$$\Phi_i^{s+} = \Phi_i^s + \frac{\Delta U_i^{dd}}{\sigma} \tag{5a}$$
$$\kappa_2 = k_I^{ext}\Phi_i^s + \frac{k_P^{ext}}{\sigma}\Delta U_i^{dd} \tag{5b}$$
$$\Delta U_i^{dd} = \tfrac{1}{N}\textstyle\sum_{k=1}^{N} U_k^{dd} - U_i^{dd} \tag{5c}$$

where $\sigma > 0$ is a scalar normalizing constant, $\Phi_i^s = \frac{\Phi_i}{\sigma}$ is the controller state vector, ΔU_i^{dd} is the controller input, and κ_2 is the controller output. The integral and proportional gains k_I^{ext} and k_P^{ext} are defined as:

$$k_I^{ext} = \sigma K_I^t, \quad k_P^{ext} = \sigma(K_P^t + K_I^t), \tag{6}$$

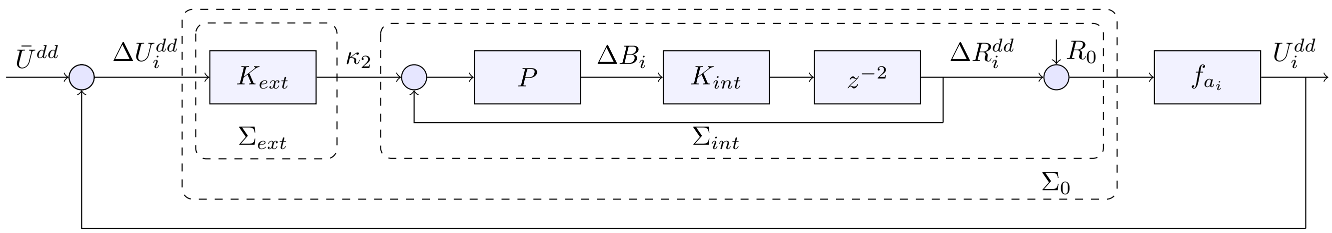


Fig. 1. Block Diagram of the controlled system

With this notation, (1e), (1f) and (1h) become:

$$\Delta R_i^{ed+} = \Delta R_i^e = -\kappa_1 \quad (7a)$$
$$\Delta R_i^{edd+} = \Delta R_i^{ed} \quad (7b)$$
$$\Delta B_i^+ = \Delta B_i + T(\Delta R_i^{edd} - \kappa_2). \quad (7c)$$

Based on (3), (7) and as represented in Figure 1, controller K_{int} performs a delayed negative feedback action over the plant through the delayed output ΔR_i^{dd}.

B. The system seen as a consensus feedback

Let $\Sigma_{ext} = (A_{ext}, B_{ext}, C_{ext}, D_{ext})$ denote the state-space representation for controller K_{ext} (*i.e.*, the system with input variable ΔU_i^{dd} and output variable κ_2), and $\Sigma_{int} = (A_{int}, B_{int}, C_{int}, D_{int})$ denote the state-space representation for the feedback loop that includes the controller K_{int} (*i.e.*, the system with input variable κ_2 and output variable ΔR_i^{edd}). Then, using (3) and (7) for Σ_{int} and (5) for Σ_{ext}, one may represent the dynamics of the i-th video stream using the states x_{int} and x_{ext} defined as:

$$x_{int} = \begin{bmatrix} \frac{\Delta B_i}{T} & \frac{\Pi_i}{T} & \Delta R_i^{ed} & \Delta R_i^{edd} \end{bmatrix}^\top, \; x_{ext} = \Phi_i \quad (8)$$

With this selection, the state-space matrices of the subsystems are given by:

$$\left(\begin{array}{c|c} A_{ext} & B_{ext} \\ \hline C_{ext} & D_{ext} \end{array}\right) = \left(\begin{array}{c|c} 1 & \frac{1}{\sigma} \\ \hline k_I^{ext} & \frac{k_P^{ext}}{\sigma} \end{array}\right) \quad (9)$$

$$\left(\begin{array}{c|c} A_{int} & B_{int} \\ \hline C_{int} & D_{int} \end{array}\right) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ -k_P^{int} & -k_I^{int} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \end{array}\right). \quad (10)$$

According to Figure 1, one can then represent the inner dynamics of each video stream, represented by Σ_0 in Figure 1 as the cascaded interconnection of Σ_{ext} and Σ_{int}, establishing the linear relation from ΔU_i^{dd} to $\Delta R_i^{dd} + R_0 = R_i^{dd}$, whose state-space representation $\Sigma_0 = (A_0, B_0, C_0, D_0)$ is such that the state matrix A_0 is lower-triangular. Actually, given the state vector $x = \begin{bmatrix} x_{ext}^\top & x_{int}^\top \end{bmatrix}^\top$, the input variable ΔU_i^{dd} and the output variable ΔR_i^{dd} we have:

$$\left(\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & D_0 \end{array}\right) = \left(\begin{array}{cc|c} A_{ext} & 0 & B_{ext} \\ B_{int}C_{ext} & A_{int} & B_{int}D_{ext} \\ \hline 0 & C_{int} & 0 \end{array}\right). \quad (11)$$

Due to its lower block triangular structure the eigenvalues of A_0 are the union of the eigenvalues of A_{int} and A_{ext}. Then the overall system dynamics is influenced by the separate actions of the two subsystems Σ_{int} and Σ_{ext}. In particular Σ_{int} performs an internal stabilizing action of each stream dynamics, and Σ_{ext} performs the external synchronization among the streams over the network.

C. Main consensus theorem

The coupling among the different video streams arises from the action of the average utility $\bar{U}^{dd}$ in (1i), acting as an input to each video stream dynamics, where the utility U_i^{dd} of each stream is a nonlinear function of the state a_i in (1a) and (1b). In particular, it is easily shown that (1g) leads to:

$$U_i^{dd} = f(a_i^{dd}, R_i^{edd}) = f(a_i^{dd}, \Delta R_i^{edd} + R_0), \quad (12)$$

so that U_i^{dd} can be seen as a nonlinear time-varying output of system Σ_0 in (11). In this paper we make the following strong assumption, so that a linear time-invariant analysis of the consensus algorithm can be performed.

Assumption 1: For each $i = 1, \ldots, N$, the input δa_i in (1a) is zero, so that a_i is constant for each i. Moreover there exist scalars h_i, $i = 1, \ldots, N$ and a scalar $K_f > 0$ such that:

$$\begin{aligned} U_i^{dd} &= f(a_i^{dd}, R_i^{edd}) = h_i + K_f R_i^{edd} \\ &= h_i + K_f R_0 + K_f \Delta R_i^{edd}, \; \forall i = 1, \ldots, N. \end{aligned} \quad (13)$$

Based on Assumption 1 and on the presence of the integral action of controller K_{ext}, we may perform a coordinate change to compensate for the action of the constant disturbance $h_i + K_f R_0$, so that the overall system can be written as an output feedback network interconnection of N identical linear systems:

$$\begin{array}{rcl} x_i^+ &=& A_0 x_i + B_0 \Delta U_i^{dd} \\ U_i^{dd} &=& K_f C_0 x_i \end{array} \quad \forall i = 1, \ldots, N \quad (14)$$

In particular, using the last equation in (5), each input ΔU_i^{dd} can be expressed, for each $i = 1, \ldots, N$, as:

$$\Delta U_i^{dd} = \bar{U}^{dd} - U_i^{dd} = \frac{1}{N} \sum_{j \neq i} U_j^{dd} - \left(1 - \frac{1}{N}\right) U_i^{dd}. \quad (15)$$

Define now the vectors $U^{dd} = \begin{bmatrix} U_1^{dd} & \ldots & U_N^{dd} \end{bmatrix}^\top$ and $\Delta U^{dd} = \begin{bmatrix} \Delta U_1^{dd} & \ldots & \Delta U_N^{dd} \end{bmatrix}^\top$. Then (15), for $i = 1, \ldots, N$, can be rewritten in the compact form:

$$\Delta U^{dd} = - \begin{bmatrix} 1-\frac{1}{N} & -\frac{1}{N} & \cdots & \cdots & -\frac{1}{N} \\ -\frac{1}{N} & 1-\frac{1}{N} & \cdots & \cdots & -\frac{1}{N} \\ \vdots & \vdots & \ddots & \vdots & \\ -\frac{1}{N} & -\frac{1}{N} & \cdots & & 1-\frac{1}{N} \end{bmatrix} U^{dd} = -LU^{dd}, \quad (16)$$

where L is the $N \times N$ *Laplacian matrix* associated with the network. The Laplacian matrix resumes the information exchanged by the subsystems. Notice that the graph related to the network described by matrix L defined in (16) is *fully connected*, *i.e.*, every vertex has an edge to every other vertex [9]. Combining (14) and (15), we obtain the following compact form for the overall system:

$$x^{+} = (I_N \otimes A_0)x + (I_N \otimes B_0)(-Ly) \quad (17a)$$

$$y = U^{dd} = K_f(I_N \otimes C_0)x, \quad (17b)$$

where y is the output representing the N utilities and $x = \left[x_1^\top \ldots x_N^\top\right]^\top$ is the overall state of the interconnected systems. Then the following theorem can be stated.

Theorem 1: Under Assumption 1, the following statements are equivalent:

(i) For any initial conditions, all utilities U_i, $i = 1, \ldots, N$ of model (3), (5), (7), (13) converge to the same value, *i.e.*, condition (2) is satisfied.

(ii) Given any solution to (17), there exists $\bar{U} \in \mathbb{R}$ such that $\lim_{j\to+\infty} y_i(j) = \bar{U}$, $\forall i = 1, \ldots, N$.

(iii) The following consensus set is uniformly globally exponentially stable for dynamics (17):

$$\mathcal{A} := \left\{x : x_i - x_j = 0, \forall i, j \in \{1, \ldots, N\}\right\} \quad (18)$$

and matrix A_{int} is Schur-Cohn.

(iv) Matrix A_{int} and matrix $A_f = A_0 - K_f\left(\frac{N}{N-1}\right)B_0C_0$ are both Schur-Cohn.

IV. Optimal tuning of the PI controllers

Item (iv) in Theorem 1 provides some useful theoretical results in order to select suboptimal controller gains, in the sense of maximizing a performance parameter, for example, the convergence rate. In particular, the selection of the optimal values may consist of two steps. First we design the inner controller K_{int} providing the explicit expression of the stability region for A_{int} as function of k_I^{int} and k_P^{int}, independently of any physical parameter. Once we fixed the inner loop control gains, the selection of the outer loop controller gains k_P^{ext} and k_I^{ext} is carried out with a similar strategy to ensure that matrix A_f in item (iv) of Theorem 1 is Schur-Cohn.

Let us first consider matrix A_{int}, which involves the gains of controller K_{int}. We want to find the constraints on the gains k_I^{int} and k_P^{int} under which the controller ensures that A_{int} be Schur-Cohn. The following lemma is proven applying the well-known Jury criterion and performing some lengthy simplifications.

Lemma 1: Matrix A_{int} in (9) is Schur-Cohn if and only if the following conditions hold:

$$k_I^{int} > 0$$

$$k_P^{int} + \frac{1-\sqrt{5}}{2} \leq k_I^{int} < k_P^{int}$$

$$(k_I^{int} - k_P^{int} - 1)^2(k_I^{int} - k_P^{int}) - (k_P^{int} + 2)(2k_I^{int} - k_P^{int}) > 0.$$

At the left of Figure 2 we show different level sets of the spectral radius $\rho(A_{int}) = \max_i\{|\lambda_i(A_{int})|\}$ as a function of parameters k_I^{int} and k_P^{int}. The bold line represents the stability limits, namely the set where $\rho(A_{int}) = 1$. Inspecting the level sets and performing a numerical optimization one obtains the optimal selection shown in Table I, that is used in the simulations of Section V.

Let us now consider matrix $A_f = A_0 - K_f\frac{N-1}{N}B_0C_0$ at item (iv) of Theorem 1, which also depends on the outer PI controller gains k_I^{ext} and k_P^{ext}. From (11), conveniently choosing $\sigma := K_f\frac{N-1}{N}$, we obtain:

$$A_f = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ -k_I^{ext} & 1 & 0 & 0 & k_P^{ext}+1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -k_P^{int} & -k_I^{int} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \quad (20)$$

Therefore, after fixing the optimized values of the internal PI loop as shown in Table I, the suboptimal parameters k_I^{ext} and k_P^{ext} are computed by a numerical procedure that minimize the spectral radius of A_f in (20). The corresponding values are reported in Table I. At the right of Figure 2 we show different level sets of the spectral radius $\rho(A_f) := \max_i\{|\lambda_i(A_f)|\}$. The bold line represents the stability limit.

Remark 1: The original system gains K_I^t, K_P^t in (1) are obtained from k_I^{ext} and k_P^{ext} using (6) and the selection $\sigma = K_f\frac{N-1}{N}$ so that the actual gains depend on the scalar K_f and the number of streams N.

V. Simulation results

To verify the performance of the control parameter design technique presented in Section IV, 6 video streams [1] of different types have been encoded during 60 s with x.264 [28] in 4CIF (704×576) format at various bit rates. The programs are Interview (Prog 1), Sport (Prog 2), Big Buck Bunny (Prog 3), Nature Documentary (Prog 4), Video Clip

[1] http://www.youtube.com/watch?v=l2Y5nIbvHLs, =G63TOHluqno, =YE7VzlLtp-4, =NNGDj9IeAuI, =rYEDA3JcQqw, =SYFFVxcRDbQ.

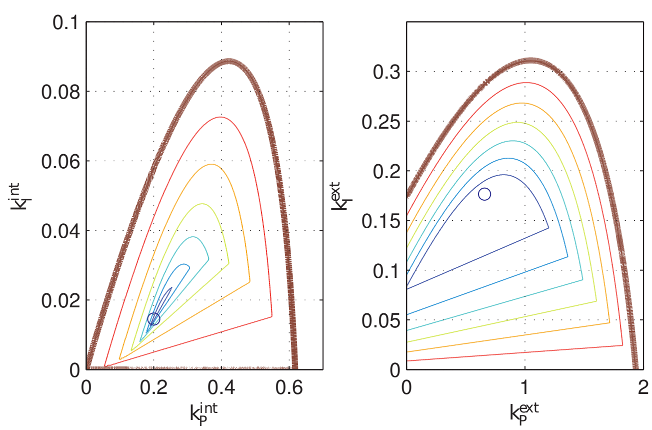


Fig. 2. Gains selection: the external line corresponds to the stability limit $\rho = 1$ and the minimum value of ρ is $\rho_{min}^{int} = 0.7964$ and $\rho_{min}^{ext} = 0.9399$, using the parameters in Table I.

k_P^{int}	k_I^{int}	k_P^{ext}	k_I^{ext}
0.2	0.0145	0.6590	0.1765

TABLE I

Optimal selection of the PI controller gains.

(Prog 5), and an extract of *Spiderman* (Prog 6). The frame rate is $F = 30$ frames/s. GoPs of 10 frames are considered, thus the GoP duration is $T = 0.33$ s. The considered utility is the Peak Signal-to-Noise Ratio (PSNR). To tune the controllers, the rate-utility characteristics of each GoP is estimated as described in [3], [4]. The control is assumed performed within the MANE, closely located to the BS to which the clients are connected. A Matlab simulation of the behavior of the servers, the network, the MANE and the clients is considered. The forward and backward propagation and queuing delays between the MANE and the servers is taken as constant and equal to T. The packets delivered by the MANE to the BS and to the clients are assumed to be well received thanks to retransmission at the MAC layer, which is not modeled here.

During the control of the streaming system, the rate-utility characteristics are *not* available at the MANE. Only the utility of the encoded packets it receives are used. They may be tagged, *e.g.*, at the RTP layer of the protocol stack. The MANE adjusts the transmission rate of each stream and provides an encoding rate target to the individual servers, which are then responsible of meeting this target by video encoding, transcoding, or bit-rate switching.

To tune the control parameters, the parameter K_f introduced in (13) has to be evaluated. For that purpose, the time and ensemble average of the linearized rate-PSNR characteristics for the four first streams have been evaluated assuming that the same constant encoding rate $R^e \in \{250, 500, 1000, 1500\}$kb/s has been used. The resulting values of K_f are $K_f(250) = 0.02$ dB/kb/s $K_f(500) = 0.01$ dB/kb/s, $K_f(1000) = 0.005$ dB/kb/s, and $K_f(1500) = 0.0033$ dB/kb/s. The product $K_f(R^e)R^e$ is almost constant for the considered experiments. A good robustness to variations of the characteristics of the video streams is obtained by taking $K_f = 0.02$ dB/kb/s. We have chosen $B_0 = 1200$ kb to tolerate significant variations of the buffering delay. With a channel rate $R_c = 4000$ kb/s and considering $N = 4$ clients, the controller parameters are $K_I^{eb} = 0.0145$ and $K_P^{eb} = 0.1855$ for the encoding rate control, whereas $K_I^t = 0.1765/\sigma$ and $K_P^t = 0.4825/\sigma$ with $\sigma = K_f N/(N-1) = 0.0267$.

Five simulation results are presented in what follows. The four video streams on which K_f has been evaluated are considered first. Figure 3(a) shows the evolution with time of the PSNR of the streams. The fairness between streams is largely improved compared to a transmission rate fair (TRF) solution illustrated in Figure 3(b). To quantify the improvement, the average absolute value of the difference of the utility (PSNR) of each stream and the average utility (PSNR) is evaluated as follows:

$$\overline{\Delta U} = \frac{1}{MN}\sum_{j=1}^{M}\sum_{k=1}^{N}\left|U_k(j) - \overline{U}(j)\right| \tag{21}$$

For the proposed tuning, $\overline{\Delta U} = 2.64$ dB, while for the TRF scheme, one gets $\overline{\Delta U} = 4.47$ dB. The results in [3] for the same scenario, were obtained with $K_I^{eb} = 0.002$, $K_P^{eb} = 0.15$, $K_I^t = 0.05$, and $K_P^t = 100$ and are represented in Figure 3(c). One gets in this case $\overline{\Delta U} = 2.96$ dB, which is larger than the results obtained in this paper. In a second scenario, the four last streams are controlled with the same control parameters to illustrate the robustness of the approach to variations of the rate-utility characteristics. Figure 4 illustrates again the evolution of the PSNR with time of the TRF scheme ($\overline{\Delta U} = 3.98$dB) and the scheme of this paper ($\overline{\Delta U} = 3.53$ dB). Due to jumps in the rate-utility characteristics of Prog. 5 (related to scene changes), reaching a very good fairness between streams is more difficult than in the previous case. Nevertheless, the control parameters designed for the first four programs still provide a satisfying behavior for the last four programs.

VI. Proof of the Main Result

A. A few technical Lemmas

In this Section we introduce a few technical lemmas establishing suitable properties of quadratic functions with respect to the following output consensus set:

$$\mathcal{A}_y := \{y : y_i - y_j = 0, \forall i, j \in \{1, \ldots, N\}\} \tag{22}$$

and the following consensus set:

$$\mathcal{A} := \{x : x_i - x_j = 0, \forall i, j \in \{1, \ldots, N\}\} \tag{23}$$

Let also recall that given a set $\mathcal{X}$, $|x|_{\mathcal{X}} = \inf_{y \in \mathcal{X}} |x - y|$. By using convexity property we can prove the following lemma. Let $\mathbf{1}_N = [1 \ \ldots \ 1]^\top \in \mathbb{R}^N$.

Lemma 2: For any pair of positive integers n, N, given set $\mathcal{A}$ in (23), we have for all $x \in \mathbb{R}^{Nn}$:

$$|x|_{\mathcal{A}}^2 = \sum_{k=1}^{N} |\bar{x} - x_k|^2, \text{ with } \bar{x} := \frac{1}{N}\sum_{k=1}^{N} x_k = \frac{1}{N}(\mathbf{1}_N^\top \otimes I_n)x, \tag{24}$$

where $x_k \in \mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$ is the (vector) average of the (vector) components of $x \in \mathbb{R}^{Nn}$.
Based on Lemma 2, we can now prove the following result.

Lemma 3: Consider any unitary matrix $T \in \mathbb{R}^{N \times N}$ whose first column is given by $\frac{1}{\sqrt{N}}\mathbf{1}_N$ and the diagonal matrix $\Delta = I_N - e_1 e_1^\top$, where $e_1 = [1 \ 0 \ \ldots \ 0]^\top \in \mathbb{R}^N$ is the first element of the Euclidean basis. Then there exist scalars $c_1, \bar{c}_1, c_2, \bar{c}_2 > 0$ such that for any $y \in \mathbb{R}^N$, the

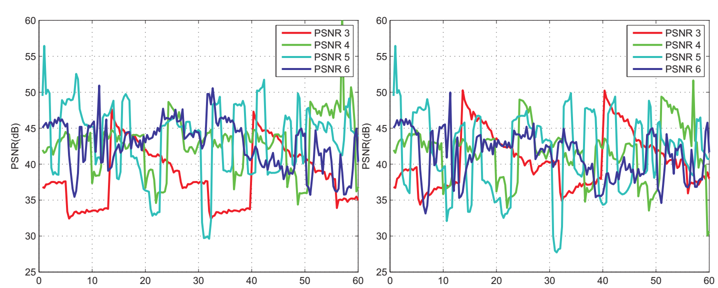
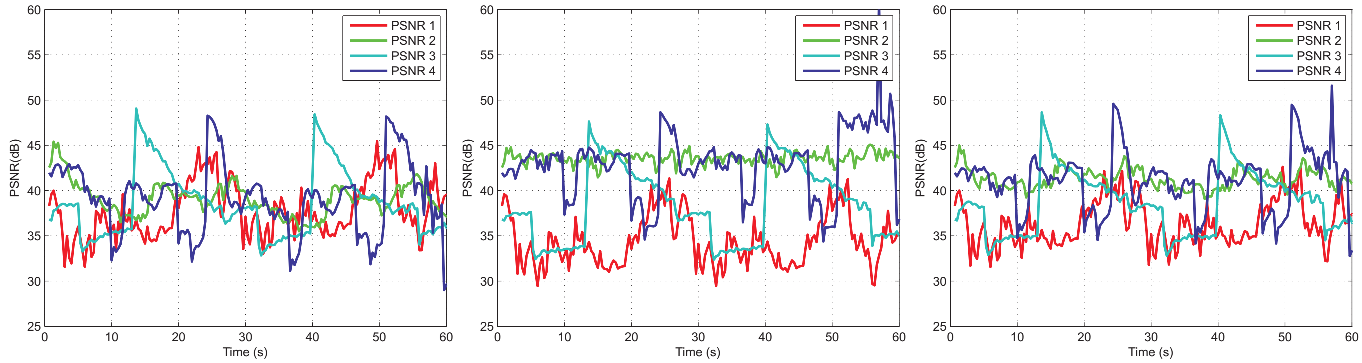


Fig. 4. PSNR of Progs 3 to 6, transmission-rate fair streaming (left) and proposed tuning of the control parameters for the utility-fair scheme (right).



(a) PSNR of Progs 1 to 4, proposed tuning of the control parameters for the utility-fair scheme.
(b) PSNR of Progs 1 to 4, transmission rate fair streaming.
(c) PSNR of Progs 1 to 4, control parameters taken from [3].

Fig. 3. PSNR of Progs 1 to 4, comparison between different control schemes.

following inequality holds:

$$\bar{c}_1 \, |y|^2_{\mathcal{A}_y} = c_1 \sum_{k=2}^{N} (y_1 - y_k)^2 \le y^\top T \Delta T^\top y \quad \text{(25a)}$$

$$\bar{c}_2 \, |y|^2_{\mathcal{A}_y} = c_2 \sum_{k=2}^{N} (y_1 - y_k)^2 \ge y^\top T \Delta T^\top y \quad \text{(25b)}$$

Moreover, for any $n \in \mathbb{N}$ and any $x \in \mathbb{R}^{Nn}$, where $x_k \in \mathbb{R}^n$, $\forall k = 1, \dots, N$, we have:

$$\bar{c}_1 \, |x|^2_{\mathcal{A}} = \sum_{k=2}^{N} |x_1 - x_k|^2 \le x^\top (T \Delta T^\top \otimes I_n) x \quad \text{(26a)}$$

$$\bar{c}_2 \, |x|^2_{\mathcal{A}} = c_2 \sum_{k=2}^{N} |x_1 - x_k|^2 \ge x^\top (T \Delta T^\top \otimes I_n) x \quad \text{(26b)}$$

Proof: Since matrix Δ has a zero in the upper left entry and ones in the remaining diagonal entries, we can write:

$$T \Delta T^\top = \bar{T} \bar{T}^\top \quad (27)$$

where $\bar{T} \in \mathbb{R}^{N \times (N-1)}$, composed by the last $N-1$ columns of T, satisfies $\bar{T}^\top \mathbf{1}_N = 0$ and has $N-1$ independent columns. Therefore, $\text{Im}\, \bar{T} \subset (\mathbf{1}_N)^\perp$. As a consequence $\text{Im} \begin{bmatrix} 1 & 1 & \dots & 1 \\ -1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{bmatrix} \subset \text{Im}\, \bar{T}$, and there exists Σ invertible such that $\Sigma \bar{T}^\top y = \tilde{y} := \begin{bmatrix} y_1 - y_2 \\ \vdots \\ y_1 - y_N \end{bmatrix} \in \mathbb{R}^{N-1}$, where $\tilde{y}$ clearly satisfies $\sum_{k=2}^{N} (y_1 - y_k)^2 = |\tilde{y}|^2$. With reference to relation (27) consider now:

$$\bar{T} \bar{T}^\top = \bar{T} \Sigma^\top \Sigma^{-\top} \Sigma^{-1} \Sigma \bar{T}^\top = \bar{T} \Sigma^\top M \Sigma \bar{T}^\top,$$

where $M = \Sigma^{-\top} \Sigma^{-1}$ is clearly positive definite. Then choosing $c_1 = \lambda_{\min}(M)$ and $c_2 = \lambda_{\max}(M)$, we may use:

$$y^\top T \Delta T^\top y = y^\top \bar{T} \bar{T}^\top y = \tilde{y}^\top M \tilde{y},$$

to obtain the inner inequalities in (25). Consider now the quadratic form:

$$x^\top (T \Delta T^\top \otimes I_n) x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}^\top (\bar{T} \bar{T}^\top \otimes I_n) \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

$$= \left((\Sigma \bar{T}^\top \otimes I_n) \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \right)^\top (M \otimes I_n) \left((\Sigma \bar{T}^\top \otimes I_n) \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} I_n & -I_n & 0 & \dots & 0 \\ I_n & 0 & -I_n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ I_n & 0 & 0 & \dots & -I_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \right)^\top$$

$$(M \otimes I_n) \left(\begin{bmatrix} I_n & -I_n & 0 & \dots & 0 \\ I_n & 0 & -I_n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ I_n & 0 & 0 & \dots & -I_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \right) = \tilde{x}^\top (M \otimes I_n) \tilde{x},$$

where $\tilde{x} = [x_1 - x_2 \; \dots \; x_1 - x_N]^\top$. Then noticing that $\lambda_{min}(M \otimes I_n) = \lambda_{min}(M) = c_1$ and $\lambda_{max}(M \otimes I_n) = \lambda_{max}(M) = c_2$ we obtain the inner inequalities in (23). To complete the proof we need to show the outer inequalities in (22), (23). To this end, it is sufficient to show that there exist positive scalars k_1 and k_2 such that for any pair n, N and any $x \in \mathbb{R}^{Nn}$, we have:

$$k_1 \, |\tilde{x}|^2 \le \sum_{k=1}^{N} |\bar{x} - x_k|^2 \le k_2 \, |\tilde{x}| \, , \quad (28)$$

and then the result follows from Lemma 2. To show (28) we first observe that $\sum_{k=1}^{N} |\bar{x} - x_k|^2 = |\bar{x} \otimes \mathbf{1}_n - x|^2$ and then the straightforward relation:

$$\tilde{x} = \underbrace{\begin{bmatrix} I_n & -I_n & 0 & \dots & 0 \\ I_n & 0 & -I_n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ I_n & 0 & 0 & \dots & -I_n \end{bmatrix}}_{T_1} (\bar{x} \otimes \mathbf{1}_n - x) \quad (29)$$

implies $|\tilde{x}|^2 = \tilde{x}^\top \tilde{x} = (\bar{x} \otimes \mathbf{1}_n - x)^\top T_1^\top T_1 (\bar{x} \otimes \mathbf{1}_n - x) \le k_1^{-1} \, |\bar{x} \otimes \mathbf{1}_n - x| \,$, where k_1^{-1} is the maximum singular value of $T_1^\top T_1$. Similarly we have:

$$\begin{aligned}
&\frac{1}{N}\begin{bmatrix}-I_n & -I_n & \dots & -I_n\end{bmatrix}\tilde{x} = \\
&= \frac{1}{N}\left(-(N-1)x_1 + \sum_{k=2}^{N} x_k + x_1 - x_1\right) \\
&= \frac{1}{N}\left(-Nx_1 + \sum_{k=1}^{N} x_k\right) = \bar{x} - x_1
\end{aligned} \tag{30}$$

which implies:

$$\begin{aligned}
&\frac{1}{N}\begin{bmatrix}(N-1)I_n & -I_n & \dots & -I_n\end{bmatrix}\tilde{x} = \\
&= \bar{x} - x_1 + \frac{N}{N}(x_1 - x_2) = \bar{x} - x_2
\end{aligned} \tag{31}$$

and, using similar reasonings:

$$(\bar{x}\otimes \mathbf{1}_n - x) = \underbrace{\begin{bmatrix} -I_n & -I_n & -I_n & \dots & -I_n \\ (N-1)I_n & -I_n & -I_n & \dots & -I_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -I_n & -I_n & -I_n & \dots & (N-1)I_n \end{bmatrix}}_{T_2}\tilde{x}, \tag{32}$$

which implies $|\bar{x}\otimes \mathbf{1}_n - x|^2 = (\bar{x}\otimes \mathbf{1}_n - x)^\top(\bar{x}\otimes \mathbf{1}_n - x) = \tilde{x}^\top T_2^\top T_2 \tilde{x} \leq k_2 |\tilde{x}|^2$, where k_2 is the maximum singular value of $T_2^\top T_2$. ∎

B. Proof of Theorem 1

Before proving Theorem 1, we introduce a general result on consensus of identical linear systems that combines the stability results in [9] with output feedback coupling analyzed in [20].

Implication (i) $\implies$ (ii) is also reported in [29, Theorem 1] for the convergence proof.

Consider N identical dynamical systems, governed by:

$$\begin{aligned} \delta x_i &= Ax_i + Bu_i \\ y_i &= Cx_i \end{aligned} \qquad i = 1,\dots,N \tag{33}$$

where $\delta x = \dot{x}$ for continuous-time and $\delta x = x^+$ for discrete-time. In (33), $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}$, $y_i \in \mathbb{R}$. Consider the interconnection:

$$u = -Ly, \tag{34}$$

where $u = [u_1 \dots u_N]^\top \in \mathbb{R}^N$, $y = [y_1 \dots y_N]^\top \in \mathbb{R}^N$ and $L = L^\top \in \mathbb{R}^{N\times N}$ is a symmetric Laplacian matrix. Also denote the eigenvalues of L as $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{N-1}$, where it is emphasized (see [9]) that L always has an eigenvalue at zero, that corresponds to the eigenvector $\mathbf{1}_N$.

Theorem 2: The following statements are equivalent:

(i) Matrices

$$A_k := A - \lambda_k BC,\ k = 1,\dots,N-1 \text{ are Schur-Cohn.} \tag{35}$$

(ii) There exists a strict quadratic Lyapunov function $V(x)$ satisfying:

$$\bar{c}_1 |x|_{\mathcal{A}}^2 \leq V(x) \leq \bar{c}_2 |x|_{\mathcal{A}}^2, \tag{36a}$$

$$\dot{V}(x)\backslash\Delta V(x) \leq -\bar{c}_3 |x|_{\mathcal{A}}^2, \tag{36b}$$

for suitable positive constants $\bar{c}_1$, $\bar{c}_2$ and $\bar{c}_3$, where $|x|_{\mathcal{A}}$ denotes the distance of x from the set $\mathcal{A}$.

(iii) The closed attractor:

$$\mathcal{A} := \{(x_1,\dots,x_N) : x_i - x_j = 0,\ \forall i,j = 1,\dots,N\} \tag{37}$$

is uniformly globally exponentially stable for the closed loop (33), (34).

(iv) The closed loop (33), (34) is such that the sub-states x_i uniformly globally exponentially synchronize to the unique solution to the following initial value problem:

$$\delta x_\circ = Ax_\circ, \quad x_\circ(0) = \frac{1}{N}\sum_{k=1}^{N} x_k(0) \tag{38}$$

Proof: We first show a preliminary transformation, then we prove the theorem in three steps: (i) $\implies$ (ii), (ii) $\implies$ (iii), (iii) $\implies$ (iv), and (iv) $\implies$ (i).

Preliminary transformation. Let us define the extended state vector $x = \begin{bmatrix}x_1^\top \dots x_N^\top\end{bmatrix}^\top$ and rewrite interconnection (33), (34) in the following compact form:

$$\delta x = (I_N \otimes A)x + (I_N \otimes B)u \tag{39}$$

$$y = (I_N \otimes C)x \tag{40}$$

$$u = -(L\otimes C)x = -(I_N \otimes C)(L\otimes I_n)x, \tag{41}$$

where $I_N \otimes A \in \mathbb{R}^{Nn\times Nn}$, $I_N \otimes B \in \mathbb{R}^{Nn\times N}$, $I_N \otimes C \in \mathbb{R}^{N\times Nn}$ and $L\otimes I_n \in \mathbb{R}^{Nn\times Nn}$. Since matrix L is symmetric, there exists a unitary matrix $T \in \mathbb{R}^{N\times N}$ (namely a matrix satisfying $T^\top T = I_N$ that diagonalizes L). In particular, let us pick T such that:

$$\Lambda = T^\top LT = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & \lambda_1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{N-1} \end{bmatrix}. \tag{42}$$

Since the upper-left entry of Λ is zero, we may select T such that its first column corresponds to the eigenvector $t_0 = \frac{1}{\sqrt{N}}\mathbf{1}_N$ associated to the zero eigenvalue $\lambda_0 = 0$ of L. Furthermore, it is easily checked that $T\otimes I_n$ transforms $L\otimes I_n$ into $\Lambda\otimes I_n$. Indeed, using the associative property of the Kronecker product we get $(T\otimes I_n)^\top(L\otimes I_n)(T\otimes I_n) = (T^\top LT\otimes I_n) = \Lambda\otimes I_n$. Let us now introduce the similarity transformation $\bar{x} = (T^\top \otimes I_n)x$. Then dynamics (39) reads:

$$\begin{aligned}
&\delta x = (T\otimes I_n)^{-1}(I_N\otimes A)(T\otimes I_n)\bar{x} + (T\otimes I_n)^{-1}(I_N\otimes B)u \\
&y = (I_N\otimes C)(T\otimes I_n)\bar{x} \\
&u = -(I_N\otimes C)(L\otimes I_n)(T\otimes I_n)\bar{x}.
\end{aligned} \tag{43}$$

Substituting in (43) the control law (third equation) into the first equation and using the associative property of the Kronecker product we obtain:

$$\delta\bar{x} = \bar{A}\bar{x}, \tag{44}$$

where the state matrix A can be computed as:

$$\begin{aligned}
\bar{A} &= (T^{-1}T\otimes A) - \\
&\quad -(T\otimes I_n)^{-1}(I_N\otimes B)(I_N\otimes C)(L\otimes I_n)(T\otimes I_n) \\
&= (I_N\otimes A) - (T^{-1}LT\otimes BC) \\
&= (I_N\otimes A) - (\Lambda\otimes BC) \\
&= (I_N\otimes A) - (I_N\otimes BC)(\Lambda\otimes I_n),
\end{aligned} \tag{45}$$

which evidently has the following block diagonal structure:

$$\bar{A} = \begin{bmatrix} A & 0 & \dots & 0 \\ 0 & A-\lambda_1 BC & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & A-\lambda_{N-1}BC \end{bmatrix} = \operatorname{diag}\{A, A_1, \dots, A_{N-1}\}, \tag{46}$$

where we use the definitions in (35).

Proof of (i) $\implies$ (ii) By assumption (35), we have that there exist matrices P_k, $k = 1, \dots, N-1$ such that:

$$A_k^\top P_k + P_k A_k = -I_n,\ k = 1, \dots, N-1\ , t \in \mathbb{R} \tag{47a}$$

$$A_k^\top P_k A_k - P_k = -I_n,\ k = 1, \dots, N-1\ , t \in \mathbb{Z} \tag{47b}$$

Construct the block diagonal matrix $\bar{P} = \operatorname{diag}\{0, P_1, \dots, P_{N-1}\}$ and define the Lyapunov function candidate:

$$V(x) = \underbrace{x^\top (T \otimes I_n)}_{\bar{x}^\top} \bar{P} \underbrace{(T^\top \otimes I_n)}_{\bar{x}} = \sum_{k=1}^{N-1} \bar{x}_k^\top P_k \bar{x}_k. \tag{48}$$

Then, from equations (44), (46) and (47) it follows that:

$$\begin{aligned} \dot{V}(x) = & \sum_{k=1}^{N-1} \bar{x}_k^\top (P_k A_k + A_k^\top P_k)\bar{x}_k = -\sum_{k=1}^{N-1} \bar{x}_k^\top \bar{x}_k, \\ \Delta V(x) = & \sum_{k=1}^{N-1} \bar{x}_k^\top (A_k^\top P_k A_k - P_k)\bar{x}_k = -\sum_{k=1}^{N-1} \bar{x}_k^\top \bar{x}_k, \end{aligned} \tag{49}$$

To prove (36), we use Lemma 2 after noticing that matrix T introduced in the preliminary step of the proof satisfies the assumption of the lemma. Then we also observe that, using matrix $\Delta = \operatorname{diag}\{0, 1, \dots, 1\}$ defined in Lemma 3, we have:

$$\begin{aligned} & \sum_{k=1}^{N-1} \bar{x}_k^2 = \bar{x}^\top (\Delta \otimes I_n)\bar{x} \\ & = ((T \otimes I_n)x)^\top (\Delta \otimes I_n)(T \otimes I_n)x \\ & = x^\top (T^\top \Delta T \otimes I_n)x \end{aligned} \tag{50}$$

Therefore, using (50), positive definiteness of P_k, $k = 1, \dots, N-1$, definition (48) and Lemma 1, we obtain:

$$\begin{aligned} V(x) & \le \underbrace{\max_{h \in \{1,\dots,N-1\}} \lambda_{\max}(P_h)}_{\bar{p}} \sum_{k=1}^{N-1} |x_k|^2 \\ & = \bar{p} x^\top (T^\top \Delta T \otimes I_n)x \le c_2 \bar{p} |x|_{\mathcal{A}}^2 \\ V(x) & \ge \underbrace{\min_{h \in \{1,\dots,N-1\}} \lambda_{\min}(P_h)}_{\underline{p}} \sum_{k=1}^{N-1} |x_k|^2 \\ & = \underline{p} x^\top (T^\top \Delta T \otimes I_n)x \ge c_1 \underline{p} |x|_{\mathcal{A}}^2, \end{aligned} \tag{51}$$

thus proving the first equation in (47) with $\bar{c}_1 = c_1 \underline{p}$ and $\bar{c}_2 = c_2 \bar{p}$. Finally, using (49), (50) and Lemma 1 we get:

$$\dot{V}(x)/\Delta V(x) \le -x^\top (T^\top \Delta T \otimes I_n)x \le -c_1 |x|_{\mathcal{A}}, \tag{52}$$

which coincides with the second equation in (36) with $\bar{c}_3 = c_1$.

Proof of (ii) $\implies$ (iii) Based on (36), UGES of $\mathcal{A}$ in (37) follows from standard Lyapunov results (see, *e.g.*, the discrete- and continuous-time special cases of the hybrid results in [25, Theorem1]).

Proof of (iii) $\implies$ (iv) Consider the dynamics of the state $x_\circ(t) := \frac{1}{N}\sum_{k=1}^{N} x_k(t)$ and note that from (33):

$$\begin{aligned} \delta x_\circ(t) & = \frac{1}{N}\sum_{k=1}^{N} \delta x_k(t) = A \sum_{k=1}^{N} x_k(t) + B \sum_{k=1}^{N} u_k(t) \\ & = A x_\circ(t) + B \underbrace{\mathbf{1}_N^\top L}_{=0} y = A x_\circ(t), \end{aligned} \tag{53}$$

where $\mathbf{1}_N^\top L = 0$ due to well known properties of Laplacian matrices. Then $x_\circ$ evolves autonomously according to (38) and corresponds to the average of states x_k. Since from (ii) $\implies$ (iii) we know that states x_k exponentially synchronize to some consensus, then they must synchronize to their average value that is $x_\circ$.

Proof of (iv) $\implies$ (i) We prove this step by contradiction. Assume that one of matrices A_k in (35) is not Schur-Cohn, and assume without loss of generality that it is A_{N-1}. Consider the coordinate system in (44) with (46). Then, from the block diagonal structure of $\bar{A}$, since A_{N-1} is not Schur-Cohn, there exists a vector $\omega^* \in \mathbb{R}^n$ (an eigenvector of one of the non-converging natural modes) such that the solution to (44) from $\bar{x}^*(0) = \begin{bmatrix} 0^\top & \dots & 0^\top & \omega^{*\top} \end{bmatrix}^\top$ corresponds to $\bar{x}^*(t) = \begin{bmatrix} 0^\top & \dots & 0^\top & \bar{x}_N^\top(t) \end{bmatrix}^\top$, where $\bar{x}_N(t)$ does not converge to zero. As a consequence, the function in (48) along this solution corresponds to:

$$V(x^*(t)) = V((T \otimes I_n)\bar{x}^*(t)) = \bar{x}_N^\top(t) P_N \bar{x}_N(t),$$

which, from linearity, remains bounded away from zero. Then, using the first inequality in (51) we have that $|x^*(t)|_{\mathcal{A}}$ is bounded away from zero, namely solution $x^*(t)$ does not converge to the consensus set. In other words, the components of $x^*(t)$ do not asymptotically synchronize, which contradicts item (iv). ■

Based on Theorem 2, the proof of Theorem 1 is given in the following.

Proof of (i) ⇔ (ii) This equivalence follows from the fact that, due to relations (14)–(16), and from the definitions in (9)–(11), model (17) coincides with the closed loop (3), (5), (7), (13).

Proof of (iii) ⇔ (iv) Applying the equivalence between items (i) and (iii) of Theorem 2 when focusing on system (17), item (iii) of Theorem 1 is equivalent to having that all eigenvalues λ_k of matrix L in (16), except for that one related to the eigenvector $\mathbf{1}_N$, are such that $A_0 - \lambda_k K_f B_0 C_0$ is Schur-Cohn. Since Laplacian matrix L has all such eigenvalues coincident and equal to $\frac{N}{N-1}$, the result trivially follows.

Proof of (iv) ⇒ (i) Similar to the previous step, this implication follows from item (iv) of Theorem 2 after noticing that system $x_\circ^+ = A x_\circ$ corresponds to system Σ_0, namely $A = A_0$, where A_0 is given in (11). Since A_{int} is Schur-Cohn by assumption, then due to its block triangular structure, matrix A_0 has a single eigenvalue at zero and all solutions to (38) converge to a constant, thereby proving item (i) of

Theorem 1.

Proof of (i) ⇒ (iv) We prove this by contradiction. Assume that item (iv) does not hold. Then either A_f is not Schur-Cohn, which implies from Theorem 2 that consensus is not achieved for some initial conditions (thereby proving that (i) does not hold), or A_f is Schur-Cohn and A_{int} is not Schur-Cohn. In this case, Theorem 2 applies because A_f is Schur-Cohn and all solutions exponentially synchronize to a solution to (38) with $A = A_0$ as in (11). Then two cases may occur:

a) A_{int} has at least one eigenvalue with magnitude larger than 1 or at least one eigenvalue on the unit circle with multiplicity larger than 1: in this case some solutions synchronize to a diverging evolution, thus item (i) does not hold;

b) A_{int} has at least one eigenvalue with magnitude 1 on the unit disk. If that eigenvalue is at 1, then due to the triangular structure, matrix A_0 has two eigenvalues in 1 (the other one coming from A_{ext}) and again some solutions synchronize to a diverging evolution. If that eigenvalue is anywhere else in the unit circle, then it generates a revolving non-constant mode and some solutions synchronize to a non-convergent oscillatory evolution.

In both cases **a)** and **b)**, item (i) does not hold and the proof is completed.

VII. Conclusion

In this paper we propose a control strategy for delivering media contents to users sharing a limited resource, ensuring quality fairness among network clients. This problem is stated in terms of consensus among identical LTI systems coupled through static output feedback. The controllers synthesis is based on a general result that provides novel analytical tools which prove the uniform global exponential synchronization among the systems outputs. Experimental results illustrate the effectiveness of the proposed control techniques. Future work may include relaxing Assumption 1 to account for the nonlinear nature of function f in (12). Furthermore, an optimization technique based on linear matrix inequalities may be developed to systematically tune the controllers gains.

Acknowledgements. The authors gratefully acknowledge N. Changuel for providing the source code for the simulations.

Appendix I
Proof of Lemma 1

The characteristic polynomial of the state matrix A_{int} is given by:

$$\lambda_{A_{int}}(z) = z^4 - 2z^3 + z^2 + k_P^{int} z + k_I^{int} - k_P^{int}. \quad (54)$$

To handle this expression we define new coefficients α and β as follows:

$$\alpha = k_P^{int}, \quad \beta = k_I^{int} - k_P^{int},$$

and polynomial (54) becomes:

$$\lambda_{A_{int}}(z) = z^4 - 2z^3 + z^2 + \alpha z + \beta. \quad (55)$$

Jury's stability criterion provides necessary and sufficient conditions on the coefficients of the polynomial (55), in order to guarantee the asymptotic stability of Σ_{int}. Applying Jury's criterion, we deduce that (55) is Schur-Cohn if and only if the parameters α and β satisfy the following constraints:

$$\alpha + \beta > 0 \quad (56a)$$
$$\beta - \alpha + 4 > 0 \quad (56b)$$
$$1 - |\beta| > 0 \quad (56c)$$
$$1 - \beta^2 - |\alpha + 2\beta| > 0 \quad (56d)$$
$$f(\alpha,\beta) > |g(\alpha,\beta)| \quad (56e)$$

where the functions $f(\alpha,\beta)$ and $g(\alpha,\beta)$ are defined as follows:

$$f(\alpha,\beta) = (\beta^2-1)^2 - (\alpha+2\beta)^2 \quad (57)$$
$$g(\alpha,\beta) = (\beta^2-1)(\beta-1) + (\alpha+2\beta)(\alpha\beta+2) \quad (58)$$

Furthermore, constraints (19) can be expressed in function of α and β:

$$\alpha + \beta > 0 \quad (59a)$$
$$\frac{1-\sqrt{5}}{2} \le \beta < 0 \quad (59b)$$
$$\beta(\beta-1)^2 - (\alpha+2\beta)(\alpha+2) > 0 \quad (59c)$$

Proving Lemma 1 is equivalent to prove that constraints (56) and (59) lead to the same solutions set.

Let $\mathcal{S}$ denote the set of points (α,β) satisfying (56), and $\mathcal{S}^*$ denote the set of points (α,β) satisfying (59). We want to prove that $\mathcal{S} \subseteq \mathcal{S}^*$ (the inclusion $\mathcal{S}^* \subseteq \mathcal{S}$ is trivial).
From constraint (56d) we get:

$$\alpha < -\beta^2 - 2\beta + 1 \le \min_\beta \left\{-\beta^2 - 2\beta + 1\right\} = 2 \quad (60a)$$
$$\alpha > \beta^2 - 2\beta - 1 \ge \max_\beta \left\{\beta^2 - 2\beta - 1\right\} = -2 \quad (60b)$$

Thus, conditions (60) and (56) imply:

$$\mathcal{S} \subseteq \mathcal{S}_1 := \{(\alpha,\beta) : |\beta| < 1, |\alpha| < 2, \alpha+\beta > 0\}$$

Moreover, it is trivial to prove the pairs $(\alpha,\beta) \in \mathcal{S}_1$ satisfy (56b). Let consider now the function $g(\alpha,\beta)$ defined in (58): $\forall\ (\alpha,\beta) \in \mathcal{S}_1$, we get the following lower bound:

$$\begin{aligned} g(\alpha,\beta) &= (\beta^2-1)(\beta-1) + \overbrace{(\alpha+\beta)}^{>0}\overbrace{(\alpha\beta+2)}^{>0} + \beta(\alpha\beta+2) \\ &> (\beta^2-1)(\beta-1) + \beta(\alpha\beta+2) \\ &= -\beta^2 + \beta^3 - \beta + 1 + \alpha\beta^2 + 2\beta \\ &> -\beta^2 + \beta^3 + \beta + 1 - \beta^3 \\ &= -\beta^2 + \beta + 1, \end{aligned}$$

and we get:

$$\tfrac{1-\sqrt{5}}{2} \le \beta \le \tfrac{1+\sqrt{5}}{2} \implies g(\alpha,\beta) > 0 \quad (61)$$

As a consequence $g(\alpha,\beta)$ is positive in the set:

$$\mathcal{S}^+ := \left\{(\alpha,\beta) : \frac{1-\sqrt{5}}{2} \le \beta < 1, |\alpha| < 2, \alpha+\beta > 0\right\}.$$

Let consider now the set $\mathcal{S}^- := \mathcal{S}_1 \setminus \mathcal{S}^+$:

$$\mathcal{S}^- = \left\{(\alpha,\beta) : -1 < \beta < \frac{1-\sqrt{5}}{2}, |\alpha| < 2, \alpha+\beta > 0\right\} \quad (62)$$

We want to prove that $\mathcal{S} \nsubseteq \mathcal{S}^-$, since inequality (56e) doesn't hold in $\mathcal{S}^-$, *i.e.*, $f(\alpha,\beta) < |g(\alpha,\beta)|$, $\forall(\alpha,\beta) \in \mathcal{S}^-$.

Two cases may occur:

a) If $g(\alpha,\beta) \geq 0$, then:

$$\begin{aligned}&f(\alpha,\beta) - g(\alpha,\beta) < \beta^4 - 2\beta^2 - \beta = \\ &= \beta(\beta+1)\left(\beta - \frac{1+\sqrt{5}}{2}\right)\left(\beta - \frac{1-\sqrt{5}}{2}\right),\end{aligned}$$

so $f(\alpha,\beta) - |g(\alpha,\beta)| = f(\alpha,\beta) - g(\alpha,\beta) < 0$ in $\mathcal{S}^-$.

b) If $g(\alpha,\beta) < 0$, then $|g(\alpha,\beta)| = -g(\alpha,\beta) > g(\alpha,\beta)$ and from the previous point we get:

$$f(\alpha,\beta) < g(\alpha,\beta) < -g(\alpha,\beta) = |g(\alpha,\beta)|\,,$$

so the statement is verified.

Moreover, considering (56e) and (61) we can conclude that $f(\alpha,\beta) > 0$ in $\mathcal{S}^+$. The function $f(\alpha,\beta)$ can be rewritten as follows:

$$f(\alpha,\beta) = -(\alpha - f_1(\beta))(\alpha - f_2(\beta))$$

where $f_1(\beta) = \beta^2 - 2\beta - 1$ and $f_2(\beta) = -\beta^2 - 2\beta + 1$. It can be easily verified that $f_1(\beta) < f_2(\beta)$ $\forall\beta : |\beta| < 1$, and the following holds:

$$f(\alpha,\beta) > 0 \Leftrightarrow \alpha < f_1(\beta) \vee \alpha > f_2(\beta). \tag{63}$$

Noticing that $f(\alpha,\beta) - g(\alpha,\beta) > 0$ holds $\forall(\alpha,\beta) \in \mathcal{S}^+$, it follows:

$$\beta\underbrace{(\beta+1)}_{>0}\underbrace{(f_1(\beta)-\alpha)}_{<0} > 0 \implies \beta < 0.$$

We want now prove that, if $f(\alpha,\beta) > 0$, (56d) is verified $\forall(\alpha,\beta) : |\beta| < 1$. In fact:

$$\begin{aligned}f(\alpha,\beta) &= (\beta^2-1)^2 - (\alpha+2\beta)^2 > 0 \\ &\implies \left|\beta^2 - 1\right| > |\alpha + 2\beta| \\ &\implies 1 - \beta^2 > |\alpha + 2\beta|\end{aligned}$$

Finally, condition (59c) can be obtained from (56e) ,(56c) and (58) as follows:

$$\begin{gathered}f(\alpha,\beta) - g(\alpha,\beta) > 0 \\ \Updownarrow \\ (\beta+1)\left[\beta(\beta-1)^2 - (\alpha+2\beta)(\alpha+2)\right] > 0 \\ \Updownarrow \\ \beta(\beta-1)^2 - (\alpha+2\beta)(\alpha+2) > 0\end{gathered}$$

We can conclude that:

$$\begin{aligned}\mathcal{S} \subseteq &\{(\alpha,\beta) : \frac{1-\sqrt{5}}{2} \leq \beta < 0, \\ &\beta(\beta-1)^2 - (\alpha+2\beta)(\alpha+2) > 0,\ \alpha+\beta > 0\} = \mathcal{S}^*.\end{aligned}$$

which concludes the proof.

Appendix II
Proof of Lemma 2

The following result is based on [24, Theorem 1.10].

Lemma 4: Given a closed, convex set $\mathcal{A} \subset \mathbb{R}^\nu$ and any vector $x \in \mathbb{R}^\nu$, there exists a unique point $y \in \mathcal{A}$ satisfying:

$$|x - y| = |x|_{\mathcal{A}} := \min_{a \in \mathcal{A}} |x - a| \tag{64}$$

Moreover, $y \in \mathcal{A}$ satisfies (64) if and only if $x \in N_{\mathcal{A}}(y)$, where:

$$N_{\mathcal{A}}(y) = \left\{n \in \mathbb{R}^\nu : \langle n - y, y - a\rangle \geq 0 \quad \forall a \in \mathcal{A}\right\} \tag{65}$$

is the normal cone to $\mathcal{A}$ at y, and y is the orthogonal projection of x onto $\mathcal{A}$ (see [18]).

Proof: We only prove the equivalence among (64) and (65) because the existence and uniqueness of y is already proven in [19, Theorem 12.3].

Proof of (65) $\Rightarrow$ (64) If $x \in N_{\mathcal{A}}(y)$ then, $\forall a \in \mathcal{A}$ we have:

$$\begin{aligned}|x-a|^2 &= |x - y + y - a|^2 \\ &= |x-y|^2 + |y-a|^2 + 2\langle x-y, y-a\rangle \\ &\geq |x-y|^2\,.\end{aligned}$$

Proof of (64)$\Rightarrow$ (65) For all $a \in \mathcal{A}$ and for any $\eta \in (0,1]$ we have form convexity that $\eta a + (1-\eta)y \in \mathcal{A}$, therefore from (64):

$$\begin{aligned}|x-y|^2 &\leq |x - (\eta a + (1-\eta)y)|^2 \\ &= |x - y - \eta(a-y)|^2 \\ &= |x-y|^2 + 2\eta\langle x-y, y-a\rangle + \eta^2\,|y-a|^2\,,\end{aligned}$$

which, dividing by η, implies:

$$2\langle x-y, y-a\rangle + \eta\,|y-a|^2 \geq 0.$$

Taking the limit as $\eta \to 0$, the statement is proven. ■

Using Lemma 4 we can prove Lemma 2. In fact, let us select $y = \mathbf{1}_N \otimes \bar{x} \in \mathbb{R}^{nN}$, so that $|x-y|^2 = \sum_{k=1}^{N} |x_k - \bar{x}|^2$. Then, according to Lemma 4, the proof is completed if $x \in N_{\mathcal{A}}(y)$. To prove this fact, first note that, since $\mathcal{A}$ is a linear subspace, for any pair of vectors $y, a \in \mathcal{A}$, we have $b := y - a \in \mathcal{A}$, so that it is enough to show:

$$\langle x - y, b\rangle \geq 0, \quad \forall b \in \mathcal{A}. \tag{66}$$

Relation (66) can be established by first noticing that $b \in \mathcal{A}$ implies that there exists $\bar{b} \in \mathbb{R}^n$ such that $b = \mathbf{1}_N \otimes \bar{b}$, and then computing:

$$\begin{aligned}\langle x-y, b\rangle &= \langle x - \mathbf{1}_N \otimes \bar{x}, \mathbf{1}_N \otimes \bar{b}\rangle \\ &= \langle \mathbf{1}_N \otimes \bar{b}, x - \mathbf{1}_N \otimes \bar{x}\rangle \\ &= (\mathbf{1}_N \otimes \bar{b})^\top \left(x - \mathbf{1}_N \otimes \frac{1}{N}(\mathbf{1}_N^\top \otimes I_n)x\right) \\ &= \frac{1}{N}(\mathbf{1}_N^\top \otimes \bar{b}^\top)\left(NI_{Nn} - \mathbf{1}_N \otimes \mathbf{1}_N^\top \otimes I_n\right)x \\ &= \frac{1}{N}(\mathbf{1}_N^\top \otimes \bar{b}^\top)\left(\left[NI_N - \mathbf{1}_N \otimes \mathbf{1}_N^\top\right] \otimes I_n\right)x \\ &= \frac{1}{N}\left(\underbrace{\mathbf{1}_N^\top\left[NI_N - \mathbf{1}_N \otimes \mathbf{1}_N^\top\right]}_{=0} \otimes \bar{b}^\top\right)x = 0\end{aligned}$$

which completes the proof.

References

[1] Cisco visual networking index: Global mobile data traffic forecast update, 2013 - 2018, January 2014. http://www.cisco.com/c/en/us/solutions/collateral/service-provider/visual-networking-index-vni/white_paper_c11-520862.pdf.

[2] N. Changuel, B. Sayadi, and M. Kieffer. Joint encoder and buffer control for statistical multiplexing of multimedia contents. In *IEEE Globecom*, pages 1–6, December 2010.

[3] N. Changuel, B. Sayadi, and M. Kieffer. Control of multiple remote servers for quality-fair delivery of multimedia contents. *IEEE Journal of Selected Areas in Communications*, January 2014. to appear.

[4] N. Changuel, B. Sayadi, and M. Kieffer. Control of distributed servers for quality-fair delivery of multiple video streams. In *Proc. ACM Multimedia*, Nara, Japan, October-November 2012.

[5] C. Chen, X. Zhu, G. de Veciana, A.C. Bovik, and Robert W. H. Jr. Rate adaptation and admission control for video transmission with subjective quality constraints. *CoRR*, abs/1311.6453, 2013.

[6] J.W Cho and S. Chong. Utility max-min flow control using slope-restricted utility functions. In *IEEE Globecom*, pages 818–824, December 2005.

[7] S. Cicalo and V. Tralli. Cross-layer algorithms for distortion-fair scalable video delivery over OFDMA wireless systems. In *IEEE Globecom*, pages 1287–1292, December 2012.

[8] J.A. Fax and R.M. Murray. Graph Laplacians and stabilization of vehicle formations. *Proceedings of the 15th IFAC World Congress on Automatic Control*, July 2002.

[9] J.A. Fax and R.M. Murray. Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control*, 49(9):1465–1476, September 2004.

[10] J.K. Hale. Diffusive coupling, dissipation, and synchronization. *Journal of Dynamics and Differential Equations*, 9(1):1–52, January 1997.

[11] A. Jadbabaie, J. Lin, and A.S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. In *IEEE Conference on Decision and Control*, volume 3, December 2002.

[12] Y. Li, Z. Li, M. Chiang, and A.R. Calderbank. Content-aware distortion-fair video streaming in congested networks. *IEEE Transactions on Multimedia*, 11(6):1182 –1193, October 2009.

[13] L. Moreau. Stability of continuous-time distributed consensus algorithms. In *IEEE 43rd Conference on Decision and Control, 2004.*, volume 4, pages 3998–4003 Vol.4, December 2004.

[14] L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, 50(2):169–182, February 2005.

[15] S. Nair and N.E. Leonard. Stable synchronization of mechanical system networks. *SIAM Journal on Control and Optimization*, 47(2):661–683, february 2008.

[16] R. Olfati-Saber and R.M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, September 2004.

[17] W. Ren. Consensus strategies for cooperative control of vehicle formations. *IET Control Theory Applications*, 1(2):505–512, March 2007.

[18] R.T. Rockafellar, R.J.B. Wets, and M. Wets. *Variational Analysis*. Grundlehren der mathematischen Wissenschaften. Springer, October 2011.

[19] W. Rudin. *Functional Analysis*. Mathematics series. McGraw-Hill, 1991.

[20] L. Scardovi and R. Sepulchre. Synchronization in networks of identical linear systems. *Automatica*, 45(11):2557–2562, November 2009.

[21] R. Sepulchre, D.A. Paley, and N.E. Leonard. Stabilization of planar collective motion: All-to-all communication. *IEEE Transactions on Automatic Control*, 52(5):811–824, May 2007.

[22] J.J.E. Slotine and W. Wang. A study of synchronization and group cooperation using partial contraction theory. In *Cooperative Control*, pages 207–228. Springer, 2005.

[23] G. B Stan and R. Sepulchre. Analysis of interconnected oscillators by dissipativity theory. *IEEE Transactions on Automatic Control*, 52(2):256–270, February 2007.

[24] S.M. Stefanov. *Separable Programming: Theory and Methods*. Applied Optimization. Springer, 2001.

[25] Andrew Teel, Fulvio Forni, and Luca Zaccarian. Lyapunov-based sufficient conditions for exponential stability in hybrid systems. *IEEE Transactions on Automatic Control*, June 2013.

[26] J. Wang, D. Cheng, and X. Hu. Consensus of multi-agent linear dynamic systems. *Asian Journal of Control*, 10(2):144–155, March 2008.

[27] P. Wieland, R. Sepulchre, and F. Allgöwer. An internal model principle is necessary and sufficient for linear output synchronization. *Automatica*, 47(5):1068–1074, July 2011.

[28] x264 Home Page. Videolan organization. http://www.videolan.org/developers/x264.html, 2011.

[29] T. Xia and L. Scardovi. Synchronization conditions for diffusively coupled linear systems. In *21st International Symposium on Mathematical Theory of Networks and Systems*, pages 1070–1075, July 2014.